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Weak and Strong Convergence Theorems for a Finite Family of Demimetric Mappings with Variational Inequality Problems in Hilbert Spaces

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Abstract. In this article, using a new nonlinear mapping called demimetric, we prove weak and strong convergence theorems for finding a common element of the set of common fixed points of a finite family of such new demimetric mappings and the set of common solutions of variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert space. Using the results, we obtain well-known and new strong convergence theorems in a Hilbert space.

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1 Introduction

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H . For a mapping $U : C \rightarrow H$, we denote by $F(U)$ the set of fixed points of U . Let k be a real number with $0 \leq k < 1$. A mapping $U : C \rightarrow H$ is called a k -strict pseudo-contraction [5] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a k -strict pseudo-contraction and $F(U) \neq \emptyset$, then we have that, for $x \in C$ and $q \in F(U)$,

$$\|Ux - q\|^2 \leq \|x - q\|^2 + k\|x - Ux\|^2.$$

From $\|Ux - q\|^2 = \|Ux - x\|^2 + \|x - q\|^2 + 2\langle Ux - x, x - q \rangle$, we have that

$$\|Ux - x\|^2 + \|x - q\|^2 + 2\langle Ux - x, x - q \rangle \leq \|x - q\|^2 + k\|x - Ux\|^2.$$

Therefore, we have that

$$2\langle x - Ux, x - q \rangle \geq (1 - k)\|x - Ux\|^2 \quad (1.1)$$

for all $x \in C$ and $q \in F(U)$. A mapping $U : C \rightarrow H$ is called generalized hybrid [10] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1 - \alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. Such a mapping U is called (α, β) -generalized hybrid. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a $(1, 0)$ -generalized hybrid mapping is nonexpansive, i.e.,

$$\|Ux - Uy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

It is nonspreading [11, 12] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Ux - Uy\|^2 \leq \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [21] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Ux - Uy\|^2 \leq \|x - y\|^2 + \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [7]. If U is generalized hybrid and $F(U) \neq \emptyset$, then we have that, for $x \in C$ and $q \in F(U)$,

$$\alpha\|q - Ux\|^2 + (1 - \alpha)\|q - Ux\|^2 \leq \beta\|q - x\|^2 + (1 - \beta)\|q - x\|^2$$

and hence $\|Ux - q\|^2 \leq \|x - q\|^2$. From this, we have that

$$2\langle x - q, x - Ux \rangle \geq \|x - Ux\|^2. \quad (1.2)$$

On the other hand, there exists such a mapping in a Banach space. Let E be a smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then, for the metric resolvent J_λ of B for $\lambda > 0$, we have from [19] that, for any $x \in E$ and $q \in B^{-1}0$,

$$\langle J_\lambda x - q, J(x - J_\lambda x) \rangle \geq 0.$$

Then we get

$$\langle J_\lambda x - x + x - q, J(x - J_\lambda x) \rangle \geq 0$$

and hence

$$\langle x - q, J(x - J_\lambda x) \rangle \geq \|x - J_\lambda x\|^2, \quad (1.3)$$

where J is the duality mapping on E . Motivated by (1.1), (1.2) and (1.3), Takahashi [23] introduced a new nonlinear mapping as follows: Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let k be a real number with $k \in (-\infty, 1)$. A mapping $U : C \rightarrow E$ with $F(U) \neq \emptyset$ is called k -demimetric if, for any $x \in C$ and $q \in F(U)$,

$$2\langle x - q, J(x - Ux) \rangle \geq (1 - k)\|x - Ux\|^2,$$

where J is the duality mapping on E . According to the definition, we get that a k -strict pseudocontraction U with $F(U) \neq \emptyset$ is k -demimetric, an (α, β) -generalized hybrid mapping U with $F(U) \neq \emptyset$ is 0-demimetric and the metric resolvent J_λ with $B^{-1}0 \neq \emptyset$ is (-1) -demimetric.

In this article, using this new nonlinear mapping called **demimetric**, we prove weak and strong convergence theorems for finding a common element of the set of common fixed points of a finite family of such new demimetric mappings and the set of common solutions of variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert space. Using the results, we obtain well-known and new strong convergence theorems in a Hilbert space.

2 Preliminaries

Throughout this paper, let \mathbb{N} be the set of positive integers and let \mathbb{R} be the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. When $\{x_n\}$ is a sequence in H , we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightarrow x$ and the weak convergence by $x_n \rightharpoonup x$. We have from [20] that for any $x, y \in H$ and $\lambda \in \mathbb{R}$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad (2.1)$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \quad (2.2)$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \quad (2.3)$$

Let C be a nonempty, closed and convex subset of a Hilbert space H . A mapping $T : C \rightarrow H$ is called **nonexpansive** if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. If $T : C \rightarrow H$ is nonexpansive, then $F(T)$ is closed and convex; see [8, 20]. For a nonempty, closed and convex subset D of H , the nearest point projection of H onto D is denoted by P_D , that is, $\|x - P_D x\| \leq \|x - y\|$ for all $x \in H$ and $y \in D$. Such a mapping P_D is called the **metric projection** of H onto D . We know that the metric projection P_D is **firmly nonexpansive**; $\|P_D x - P_D y\|^2 \leq \langle P_D x - P_D y, x - y \rangle$ for all $x, y \in H$. Furthermore, $\langle x - P_D x, y - P_D x \rangle \leq 0$ holds for all $x \in H$ and $y \in D$; see [18, 20]. Using this inequality and (2.3), we have that

$$\|P_D x - y\|^2 + \|P_D x - x\|^2 \leq \|x - y\|^2, \quad \forall x \in H, y \in D. \quad (2.4)$$

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . For $\alpha > 0$, a mapping $A : C \rightarrow H$ is called **α -inverse strongly monotone** if

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

If A is α -inverse-strongly monotone and $0 < \lambda \leq 2\alpha$, then $I - \lambda A : C \rightarrow H$ is nonexpansive. In fact, we have that for all $x, y \in C$,

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|x - y - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay \rangle + \lambda^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\lambda\alpha\|Ax - Ay\|^2 + \lambda^2\|Ax - Ay\|^2 \\ &= \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

Thus, $I - \lambda A : C \rightarrow H$ is nonexpansive; see [1, 16, 20] for more results of inverse-strongly monotone mappings. The variational inequality problem for $A : C \rightarrow H$ is to find a point $u \in C$ such that

$$\langle Au, x - u \rangle \geq 0, \quad \forall x \in C. \quad (2.5)$$

The set of solutions of (2.5) is denoted by $VI(C, A)$. We also have that, for any $\lambda > 0$, $u = P_C(I - \lambda A)u$ if and only if $u \in VI(C, A)$. In fact, let $\lambda > 0$. Then, for $u \in C$,

$$\begin{aligned} u = P_C(I - \lambda A)u &\iff \langle (I - \lambda A)u - u, u - y \rangle \geq 0, \quad \forall y \in C \\ &\iff \langle -\lambda Au, u - y \rangle \geq 0, \quad \forall y \in C \\ &\iff \langle Au, u - y \rangle \leq 0, \quad \forall y \in C \\ &\iff \langle Au, y - u \rangle \geq 0, \quad \forall y \in C \\ &\iff u \in VI(C, A). \end{aligned}$$

In the case when a Banach space E is a Hilbert space, the definition of a demimetric mapping is as follows: Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $k \in (-\infty, 1)$. A mapping $U : C \rightarrow H$ with $F(U) \neq \emptyset$ is called k -demimetric if, for any $x \in C$ and $q \in F(U)$,

$$2\langle x - q, x - Ux \rangle \geq (1 - k)\|x - Ux\|^2.$$

The following lemma which was essentially proved in [23] is important and crucial in the proof of our main result. For the sake of completeness, we give the proof.

Lemma 2.1 ([23]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let k be a real number with $k \in (-\infty, 1)$ and let U be a k -demimetric mapping of C into H . Then $F(U)$ is closed and convex.*

Proof. Let us show that $F(U)$ is closed. For a sequence $\{q_n\}$ such that $q_n \rightarrow q$ and $q_n \in F(U)$, we have from the definition of U that

$$2\langle q - q_n, q - Uq \rangle \geq (1 - k)\|q - Uq\|^2.$$

From $q_n \rightarrow q$, we have $0 \geq (1 - k)\|q - Uq\|^2$. From $1 - k > 0$, we have $\|q - Uq\|^2 = 0$ and hence $q = Uq$. This implies that $F(U)$ is closed.

Let us prove that $F(U)$ is convex. Let $p, q \in F(U)$ and set $x = \alpha p + (1 - \alpha)q$, where $\alpha \in [0, 1]$. Then we have

$$\begin{aligned} 2\|x - Ux\|^2 &= 2\langle x - Ux, x - Ux \rangle \\ &= 2\langle \alpha p + (1 - \alpha)q - Ux, x - Ux \rangle \\ &= 2\langle \alpha p + (1 - \alpha)q - (\alpha Ux + (1 - \alpha)Ux), x - Ux \rangle \\ &= 2\alpha\langle p - Ux, x - Ux \rangle + 2(1 - \alpha)\langle q - Ux, x - Ux \rangle \\ &= 2\alpha\langle p - x + x - Ux, x - Ux \rangle + 2(1 - \alpha)\langle q - x + x - Ux, x - Ux \rangle \\ &\leq \alpha(k - 1)\|x - Ux\|^2 + 2\alpha\|x - Ux\|^2 \\ &\quad + (1 - \alpha)(k - 1)\|x - Ux\|^2 + 2(1 - \alpha)\|x - Ux\|^2 \\ &= (k - 1)\|x - Ux\|^2 + 2\|x - Ux\|^2 \end{aligned}$$

and hence $0 \leq (k - 1)\|x - Ux\|^2$. We have from $0 > k - 1$ that $\|x - Ux\| \leq 0$ and hence $x = Ux$. This means that $F(U)$ is convex. \square

The following lemma is used in the proof of our main result.

Lemma 2.2 ([26]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $k \in (-\infty, 1)$ and let T be a k -demimetric mapping of C into H such that $F(T)$ is nonempty. Let λ be a real number with $0 < \lambda \leq 1 - k$ and define $S = (1 - \lambda)I + \lambda T$. Then S is a quasi-nonexpansive mapping of C into H .*

Proof. It is obvious that $F(T) = F(S)$. Since T be a k -demimetric mapping of C into H , we have that for any $x \in C$ and $z \in F(S)$,

$$\begin{aligned} 2\langle x - z, x - Sx \rangle &= 2\langle x - z, x - (1 - \lambda)x - \lambda Tx \rangle = 2\lambda\langle x - z, x - Tx \rangle \\ &\geq \lambda(1 - k)\|x - Tx\|^2 = \lambda^2 \frac{1 - k}{\lambda} \|x - Tx\|^2 \\ &= \frac{1 - k}{\lambda} \|\lambda x - \lambda Tx\|^2 = \frac{1 - k}{\lambda} \|x - Sx\|^2 \\ &\geq \frac{\lambda}{\lambda} \|x - Sx\|^2 = \|x - Sx\|^2. \end{aligned}$$

Then S is a 0-demimetric mapping. Furthermore, we have from (2.3) that for any $x \in C$ and $z \in F(S)$,

$$\begin{aligned} \|x - Sx\|^2 &\leq 2\langle x - z, x - Sx \rangle \\ &\iff \|x - Sx\|^2 \leq \|x - Sx\|^2 + \|x - z\|^2 - \|Sx - z\|^2 \\ &\iff \|Sx - z\|^2 \leq \|x - z\|^2 \\ &\iff \|Sx - z\| \leq \|x - z\|. \end{aligned}$$

Therefore, S is quasi-nonexpansive. □

3 Main Results

In this section, we first prove a weak convergence theorem of Mann's type iteration for finding a common element of the set of common fixed points for a finite family of demimetric mappings and the set of common solutions of variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert space. Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . A mapping $U : C \rightarrow H$ is called demiclosed if, for a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup w$ and $x_n - Ux_n \rightarrow 0$, $w = Uw$ holds. For example, if C is a nonempty, closed and convex subset of H and T is a nonexpansive mapping of C of H , then T is demiclosed; see [20].

Theorem 3.1 ([13]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $\{k_1, \dots, k_M\} \subset (-\infty, 1)$ and $\{\mu_1, \dots, \mu_N\} \subset (0, \infty)$. Let $\{T_j\}_{j=1}^M$ be a finite family of k_j -demimetric and demiclosed mappings of C into H and let $\{B_i\}_{i=1}^N$ be a finite family of μ_i -inverse strongly monotone mappings of C into H . Assume that*

$$\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i)) \neq \emptyset.$$

For any $x_1 = x \in C$, define $\{x_n\}$ as follows:

$$\begin{cases} z_n = \sum_{j=1}^M \xi_j((1 - \lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^N \sigma_i P_C(I - \eta_n B_i)x_n, \\ x_{n+1} = P_C(\alpha_n x_n + \beta_n z_n + \gamma_n w_n), \end{cases}$$

where $\{\lambda_n\}, \{\eta_n\} \subset (0, \infty)$, $\{\xi_1, \dots, \xi_M\}, \{\sigma_1, \dots, \sigma_N\} \subset (0, 1)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ and $a, b, c \in \mathbb{R}$ satisfy the following conditions:

- (1) $0 < a \leq \lambda_n \leq \min\{1 - k_1, \dots, 1 - k_M\}$, $0 < b \leq \eta_n \leq 2 \min\{\mu_1, \dots, \mu_N\}$;
- (2) $\sum_{j=1}^M \xi_j = 1$ and $\sum_{i=1}^N \sigma_i = 1$;
- (3) $0 < c \leq \alpha_n, \beta_n, \gamma_n < 1$ and $\alpha_n + \beta_n + \gamma_n = 1$.

Then the sequence $\{x_n\}$ converges weakly to a point $z_0 \in \cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i))$, where $z_0 = \lim_{n \rightarrow \infty} P_{\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i))} x_n$.

Next, we prove a strong convergence theorem of Halpern's type iteration for finding a common element of the set of common fixed points for a finite family of demimetric mappings and the set of common solutions of variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert space.

Theorem 3.2 ([24]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $\{k_1, \dots, k_M\} \subset (-\infty, 1)$ and $\{\mu_1, \dots, \mu_N\} \subset (0, \infty)$. Let $\{T_j\}_{j=1}^M$ be a finite family of k_j -demimetric and demiclosed mappings of C into H and let $\{B_i\}_{i=1}^N$ be a finite family of μ_i -inverse strongly monotone mappings of C into H . Assume that*

$$\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i)) \neq \emptyset.$$

Let $\{u_n\}$ be a sequence in C such that $u_n \rightarrow u$. For $x_1 = x \in C$, let $\{x_n\} \subset C$ be a sequence generated by

$$\begin{cases} z_n = \sum_{j=1}^M \xi_j((1 - \lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^N \sigma_i P_C(I - \eta_n B_i)x_n, \\ x_{n+1} = \delta_n u_n + (1 - \delta_n)(P_C(\alpha_n x_n + \beta_n z_n + \gamma_n w_n)), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\}, \{\eta_n\} \subset (0, \infty)$, $\{\xi_1, \dots, \xi_M\}, \{\sigma_1, \dots, \sigma_N\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0, 1)$ and $a, b, c \in \mathbb{R}$ satisfy the following conditions:

- (1) $0 < a \leq \lambda_n \leq \min\{1 - k_1, \dots, 1 - k_M\}$, $0 < b \leq \eta_n \leq 2 \min\{\mu_1, \dots, \mu_N\}$;
- (2) $\sum_{j=1}^M \xi_j = 1$ and $\sum_{i=1}^N \sigma_i = 1$;
- (3) $0 < c \leq \alpha_n, \beta_n, \gamma_n < 1$ and $\alpha_n + \beta_n + \gamma_n = 1$;
- (4) $\lim_{n \rightarrow \infty} \delta_n = 0$ and $\sum_{i=1}^{\infty} \delta_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i))$, where $z_0 = P_{\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i))} u$.

Using the hybrid method by Nakajo and Takahashi [17], we can also prove a strong convergence theorem for finding a common element of the set of common fixed points for a finite family of demimetric mappings and the set of common solutions of variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert space.

Theorem 3.3 ([2]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $\{k_1, \dots, k_M\} \subset (-\infty, 1)$ and $\{\mu_1, \dots, \mu_N\} \subset (0, \infty)$. Let $\{T_j\}_{j=1}^M$ be a finite*

family of k_j -demimetric and demiclosed mappings of C into H and let $\{B_i\}_{i=1}^N$ be a finite family of μ_i -inverse strongly monotone mappings of C into H . Assume that

$$\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i)) \neq \emptyset.$$

Let $x_1 \in C$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = \sum_{j=1}^M \xi_j((1 - \lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^N \sigma_i P_C(I - \eta_n B_i)x_n, \\ y_n = \alpha_n x_n + \beta_n z_n + \gamma_n w_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\}, \{\eta_n\} \subset (0, \infty)$, $\{\xi_1, \dots, \xi_M\}, \{\sigma_1, \dots, \sigma_N\} \subset (0, 1)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ and $a, b, c \in \mathbb{R}$ satisfy the following conditions:

- (1) $0 < a \leq \lambda_n \leq \min\{1 - k_1, \dots, 1 - k_M\}$, $0 < b \leq \eta_n \leq 2 \min\{\mu_1, \dots, \mu_N\}$;
- (2) $\sum_{j=1}^M \xi_j = 1$ and $\sum_{i=1}^N \sigma_i = 1$;
- (3) $0 < c \leq \alpha_n, \beta_n, \gamma_n < 1$ and $\alpha_n + \beta_n + \gamma_n = 1$.

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i))$, where $z_0 = P_{\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i))} x_1$.

Using the shrinking projection method [25], we finally prove a strong convergence theorem for finding a common element of the set of common fixed points for a finite family of demimetric mappings and the set of common solutions of variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert space.

Theorem 3.4 ([26]). Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $\{k_1, \dots, k_M\} \subset (-\infty, 1)$ and $\{\mu_1, \dots, \mu_N\} \subset (0, \infty)$. Let $\{T_j\}_{j=1}^M$ be a finite family of k_j -demimetric and demiclosed mappings of C into H and let $\{B_i\}_{i=1}^N$ be a finite family of μ_i -inverse strongly monotone mappings of C into H . Assume that

$$\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i)) \neq \emptyset.$$

Let $x_1 \in C$ and $C_1 = C$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = \sum_{j=1}^M \xi_j((1 - \lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^N \sigma_i P_C(I - \eta_n B_i)x_n, \\ y_n = \alpha_n x_n + \beta_n z_n + \gamma_n w_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\}, \{\eta_n\} \subset (0, \infty)$, $\{\xi_1, \dots, \xi_M\}, \{\sigma_1, \dots, \sigma_N\} \subset (0, 1)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ and $a, b, c \in \mathbb{R}$ satisfy the following conditions:

- (1) $0 < a \leq \lambda_n \leq \min\{1 - k_1, \dots, 1 - k_M\}$, $0 < b \leq \eta_n \leq 2 \min\{\mu_1, \dots, \mu_N\}$;
- (2) $\sum_{j=1}^M \xi_j = 1$ and $\sum_{i=1}^N \sigma_i = 1$;
- (3) $0 < c \leq \alpha_n, \beta_n, \gamma_n < 1$ and $\alpha_n + \beta_n + \gamma_n = 1$.

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i))$, where $z_0 = P_{\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i))} x_1$.

4 Applicationss

In this section, we apply Theorems 3.1, 3.2, 3.3 and 3.4 to obtain well-known and new strong convergence theorems in Hilbert spaces. We know the following lemmas obtained by Marino and Xu [15] and Kocourek, Takahashi and Yao [10]; see also [27, 28].

Lemma 4.1 ([15, 27]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let k be a real number with $0 \leq k < 1$ and $U : C \rightarrow H$ be a k -strict pseudo-contraction. If $x_n \rightarrow z$ and $x_n - Ux_n \rightarrow 0$, then $z \in F(U)$.*

Lemma 4.2 ([10, 28]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $U : C \rightarrow H$ be generalized hybrid. If $x_n \rightarrow z$ and $x_n - Ux_n \rightarrow 0$, then $z \in F(U)$.*

Using Theorem 3.1, we obtain the following weak convergence results.

Theorem 4.3. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $\{\mu_1, \dots, \mu_N\} \subset (0, \infty)$. Let $\{B_i\}_{i=1}^N$ be a finite family of μ_i -inverse strongly monotone mappings of C into H . Assume that $\cap_{i=1}^N VI(C, B_i) \neq \emptyset$. For any $x_1 = x \in C$, define $\{x_n\}$ as follows:*

$$\begin{cases} w_n = \sum_{i=1}^N \sigma_i P_C(I - \eta_n B_i)x_n, \\ x_{n+1} = \alpha_n x_n + \gamma_n w_n, \end{cases}$$

where $\{\eta_n\} \subset (0, \infty)$, $\{\sigma_1, \dots, \sigma_N\} \subset (0, 1)$, $\{\alpha_n\}, \{\gamma_n\} \subset (0, 1)$ and $b, c \in \mathbb{R}$ satisfy the following conditions:

- (1) $0 < b \leq \eta_n \leq 2 \min\{\mu_1, \dots, \mu_N\}$;
- (2) $\sum_{i=1}^N \sigma_i = 1$;
- (3) $0 < c \leq \alpha_n, \gamma_n < 1$ and $\alpha_n + \gamma_n = 1$.

Then $\{x_n\}$ converges weakly to $z_0 \in \cap_{i=1}^N VI(C, B_i)$, where $z_0 = \lim_{n \rightarrow \infty} P_{\cap_{i=1}^N VI(C, B_i)} x_n$.

Proof. The identity mapping I is a $\frac{1}{2}$ -demimetric mapping of C into H . Putting $T_j = I$ for all $j \in \{1, \dots, M\}$ and $\lambda_n = \frac{1}{2}$ for all $n \in \mathbb{N}$ in Theorem 3.1, we have that $z_n = x_n$ for all $n \in \mathbb{N}$. Furthermore, replacing $\beta_n + \gamma_n$ by γ_n , we have the desired result from Theorem 3.1. \square

Theorem 4.4. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $\{T_j\}_{j=1}^M$ be a finite family of generalized hybrid mappings of C into H and let $\{U_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into H . Assume that*

$$\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N F(U_i)) \neq \emptyset.$$

For any $x_1 = x \in C$, define $\{x_n\}$ as follows:

$$\begin{cases} z_n = \sum_{j=1}^M \xi_j ((1 - \lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^N \sigma_i P_C((1 - \eta_n)I + \eta_n U_i)x_n, \\ x_{n+1} = P_C(\alpha_n x_n + \beta_n z_n + \gamma_n w_n), \end{cases}$$

where $\{\lambda_n\}, \{\eta_n\} \subset (0, \infty)$, $\{\xi_1, \dots, \xi_M\}, \{\sigma_1, \dots, \sigma_N\} \subset (0, 1)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ and $a, b, c \in \mathbb{R}$ satisfy the following conditions:

- (1) $0 < a \leq \lambda_n \leq 1$, $0 < b \leq \eta_n \leq 1$;
- (2) $\sum_{j=1}^M \xi_j = 1$ and $\sum_{i=1}^N \sigma_i = 1$;
- (3) $0 < c \leq \alpha_n, \beta_n, \gamma_n < 1$ and $\alpha_n + \beta_n + \gamma_n = 1$.

Then the sequence $\{x_n\}$ converges weakly to a point $z_0 \in \cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N F(U_i))$, where $z_0 = \lim_{n \rightarrow \infty} P_{\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N F(U_i))} x_n$.

Proof. Since T_j is generalized hybrid, T_j is 0-demimetric. Furthermore, from Lemma 4.2 T_j is demiclosed. Since U_i is nonexpansive, $B_i = I - U_i$ is a $\frac{1}{2}$ -inverse strongly monotone mapping. We also have from $\cap_{i=1}^N F(U_i) \neq \emptyset$ that

$$\cap_{i=1}^N VI(C, I - U_i) = \cap_{i=1}^N F(P_C U_i) = \cap_{i=1}^N F(U_i).$$

Therefore, we have the desired result from Theorem 3.1. \square

Using Theorem 3.2, we can prove a strong convergence theorem for a finite family of strict pseudo-contractions in a Hilbert space.

Theorem 4.5. Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $\{k_1, \dots, k_M\} \subset [0, 1)$ and let $\{T_j\}_{j=1}^M$ be a finite family of k_j -strict pseudo-contractions of C into H . Let $\{u_n\}$ be a sequence in C such that $u_n \rightarrow u$. Assume that $\cap_{j=1}^M F(T_j) \neq \emptyset$. For any $x_1 = x \in C$, define $\{x_n\}$ as follows:

$$\begin{cases} z_n = \sum_{j=1}^M \xi_j ((1 - \lambda_n)I + \lambda_n T_j) x_n, \\ x_{n+1} = \delta_n u_n + (1 - \delta_n) (P_C (\alpha_n x_n + \beta_n z_n)), \end{cases}$$

where $a, c \in \mathbb{R}$, $\{\lambda_n\} \subset (0, \infty)$, $\{\xi_1, \dots, \xi_M\} \subset (0, 1)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ satisfy the following conditions:

- (1) $0 < a \leq \lambda_n \leq \min\{1 - k_1, \dots, 1 - k_M\}$;
- (2) $\sum_{j=1}^M \xi_j = 1$;
- (3) $0 < c \leq \alpha_n, \beta_n < 1$ and $\alpha_n + \beta_n = 1$;
- (4) $\lim_{n \rightarrow \infty} \delta_n = 0$ and $\sum_{i=1}^{\infty} \delta_n = \infty$.

Then $\{x_n\}$ converges strongly to $z_0 \in \cap_{j=1}^M F(T_j)$, where $z_0 = P_{\cap_{j=1}^M F(T_j)} u$.

Proof. Since T_j is a k_j -strict pseudo-contraction of C into H such that $F(T_j) \neq \emptyset$, T_j is k_j -demimetric. Furthermore, from Lemma 4.1, T_j is demiclosed. Furthermore, if $B_i = 0$ for all $i \in \{1, \dots, N\}$ in Theorem 3.2, then B_i is a 1-inverse strongly monotone mapping. Putting $\eta_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.2, we have that $w_n = x_n$ for all $n \in \mathbb{N}$. Furthermore, replacing $\beta_n + \gamma_n$ by β_n , we have the desired result from Theorem 3.2. \square

Using Theorem 3.3, we prove a strong convergence theorem for a finite family of inverse strongly monotone mappings in a Hilbert space.

Theorem 4.6. Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $\{\mu_1, \dots, \mu_N\} \subset (0, \infty)$. Let $\{B_i\}_{i=1}^N$ be a finite family of μ_i -inverse strongly monotone mappings of C into H . Assume that $\cap_{i=1}^N VI(C, B_i) \neq \emptyset$. Let $x_1 \in C$. Let $\{x_n\}$ be a sequence

generated by

$$\begin{cases} w_n = \sum_{i=1}^N \sigma_i P_C(I - \eta_n B_i)x_n, \\ y_n = \alpha_n x_n + \gamma_n w_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $b, c \in \mathbb{R}$, $\{\eta_n\} \subset (0, \infty)$, $\{\sigma_1, \dots, \sigma_N\} \subset (0, 1)$ and $\{\alpha_n\}, \{\gamma_n\} \subset (0, 1)$ satisfy the following conditions:

- (1) $0 < b \leq \eta_n \leq 2 \min\{\mu_1, \dots, \mu_N\}$;
- (2) $\sum_{i=1}^N \sigma_i = 1$;
- (3) $0 < c \leq \alpha_n, \gamma_n < 1$ and $\alpha_n + \gamma_n = 1$.

Then $\{x_n\}$ converges strongly to $z_0 \in \cap_{i=1}^N VI(C, B_i)$, where $z_0 = P_{\cap_{i=1}^N VI(C, B_i)} x_1$.

Proof. The identity mapping I is a $\frac{1}{2}$ -demimetric mapping of C into H . Putting $T_j = I$ for all $j \in \{1, \dots, M\}$ and $\lambda_n = \frac{1}{2}$ for all $n \in \mathbb{N}$ in Theorem 3.3, we have that $z_n = x_n$ for all $n \in \mathbb{N}$. Furthermore, replace $\beta_n + \gamma_n$ by γ_n . Thus, we have the desired result from Theorem 3.3. \square

Using Theorem 3.4, we prove a strong convergence theorem for a finite family of generalized hybrid mappings and a finite family of inverse strongly monotone mappings in a Hilbert space.

Theorem 4.7. Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $\{\mu_1, \dots, \mu_N\} \subset (0, \infty)$. Let $\{T_j\}_{j=1}^M$ be a finite family of generalized hybrid mappings of C into H and let $\{B_i\}_{i=1}^N$ be a finite family of μ_i -inverse strongly monotone mappings of C into H . Assume that

$$\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i)) \neq \emptyset.$$

Let $x_1 \in C$ and $C_1 = C$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = \sum_{j=1}^M \xi_j ((1 - \lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^N \sigma_i P_C(I - \eta_n B_i)x_n, \\ y_n = \alpha_n x_n + \beta_n z_n + \gamma_n w_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\}, \{\eta_n\} \subset (0, \infty)$, $\{\xi_1, \dots, \xi_M\}, \{\sigma_1, \dots, \sigma_N\} \subset (0, 1)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ and $a, b, c \in \mathbb{R}$ satisfy the following conditions:

- (1) $0 < a \leq \lambda_n \leq 1$, $0 < b \leq \eta_n \leq 2 \min\{\mu_1, \dots, \mu_N\}$;
- (2) $\sum_{j=1}^M \xi_j = 1$ and $\sum_{i=1}^N \sigma_i = 1$;
- (3) $0 < c \leq \alpha_n, \beta_n, \gamma_n < 1$ and $\alpha_n + \beta_n + \gamma_n = 1$.

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i))$, where $z_0 = P_{\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i))} x_1$.

Proof. Since T_j is a generalized hybrid mapping of C into H such that $F(T_j) \neq \emptyset$, from (1.2), T_j is 0-demimetric. Furthermore, from Lemma 4.2, T_j is demiclosed. Therefore, we have the desired result from Theorem 3.4. \square

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